15-122: Principles of Imperative Computation

Recitation 12b Solutions

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Practice!

Unbounded array insertion — aggregate analysis

Using aggregate analysis, show that adding an element to an unbounded array takes amortized $O(1)$ time. We'll only count the number of array writes, for simplicity.

Solution: Consider n insertions to an array with limit n (the array starts out empty).

Exactly one insertion will take n steps: $n-1$ to copy over all $n-1$ elements from the small array to the large array and one to insert the new element.

Every other insertion takes 1 step — we just insert the element.

So, over n insertions, we have a total of $(n-1)*1+1*(n) = 2n-1$ steps.

$$\frac{2n-1}{n} = 2 - \frac{1}{n} \in O(1)$$

Thus, we have an amortized runtime of $O(1)$ for insertion.

Unbounded array insertion — accounting analysis

Using an accounting analysis, show that adding an element to an unbounded array takes amortized $O(1)$ time (the array starts out empty).

Again, we'll only be counting array writes.

Assume that the limit is n, and that $n > 0$ (the analysis can also work when $n = 0$, but there's an annoying special case).

Solution: When we insert, we need to pay 1 token (this reflects the cost of inserting into an array). We can also put two tokens aside. So, we lose 3 tokens for every insertion. Note that this is still a constant cost.

Then, when it comes time to make a new array we can reach into our stash of coins, which now has $2(n-1)$ tokens, since we've inserted into the array $n-1$ times and put 2 tokens into the stash for each insertion. To help pay for the cost of copying elements over, we grab $n-1$ tokens from the stash and use them. Now our stash has $n-1$ tokens left in it. Then, we pay 1 token to insert the new element in the array, and set two tokens aside in the stash (leaving us with $n+1$ tokens).

On future resizings of the array, we'll have enough tokens since we insert two tokens every time — one of them will pay to copy the element that was inserted when we got it, and the other will pay for one in the first half of the array.

In this way, we pay 3 tokens for every insert, which is a constant. So, insertion into the unbounded array is constant time.

Binary counter

Consider the situation where we have an n-bit binary number. Assume that flipping a bit (changing it

from 1 to 0, or from 0 to 1) is a constant-time operation.

What is the amortized time complexity of incrementing the number, in terms of n?

Solution: Consider incrementing a number repeatedly. We'll flip the lowest bit every time we increment, the second bit every other time we increment, and so on. More generally, we flip bit i every $\frac{1}{2^i}$ increments.

To go from $00\ldots00$ to $11\ldots11$, we would need $2^n - 1$ increments.

We'll flip bit i $\frac{2^n}{2^i} - 1$ times. (To be formal, we'd want to prove this, but we'll be a little loose about it for now.)

Thus, we have a total of

$$\sum_{i=0}^{n} \left(\frac{2^n}{2^i} - 1 \right) = \left(\sum_{i=0}^{n} \frac{2^n}{2^i} \right) - n = 2^{n+1} - 2 - n$$

flips. However, this was over $2^n - 1$ operations, so we divide and see:

$$\frac{2^{n+1} - 2 - n}{2^n - 1} = \frac{2(2^n - 1)}{2^n - 1} - \frac{n}{2^n - 1} = 2 - \frac{n}{2^n - 1} \in O(1)$$

We can also use an accounting analysis to show that n increments will cost $O(n)$ tokens. (And so each increment costs $O(1)$ token.)

Start by putting a token next to each bit. There are n bits, so this costs n tokens to start. Then, for each increment operation, we pay 1 tokens. For every bit that must flip from 0 to 1, we simply use the token sitting next to it to pay for the flip. Then, for the one bit that must flip from 1 to 0, we pay for that using one token and put a token down next to that bit to pay for later operations.

In this way, we only need to contribute 2 new tokens for each increment operation, and our n increments only cost $n + 2n \in O(n)$ tokens. Thus, each increment has an amortized cost of $O(1)$ tokens.